Downlink Coverage Probability in MIMO HetNets

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***Abstract*—The growing popularity of small cells is driving cellular networks of yesterday towards heterogeneity and randomness. Soon, hundreds of unplanned user deployed femtocells and tens of operator managed picocells will coexist in a typical macrocell. One of the natural ways to model base station (BS) locations in such heterogeneous cellular networks, or HetNets, is by using random spatial models. While sufficient progress has been made in modeling single-antenna HetNets, our focus in this paper is on multi-antenna HetNets for which the modeling tools are not well developed. Assuming K classes of BSs, which may differ in terms of transmit power, target signal to interference ratio (SIR), deployment density, number of transmit antennas and multi-antenna technique, we derive an upper bound on the coverage probability using tools from stochastic geometry. We show that the bound can be reduced to closed form in certain cases of interest and is tight down to very low target SIRs.**

I. Introduction

A potentially undesirable effect of small cells deployments is the increasing randomness in the base station (BS) locations due to lack of site planning. The departure of cellular networks from regular deployments of yesterday towards nearly random deployments of today means that the cellular models that were based on the regularity assumption of BS locations, e.g., deterministic grid model, are no longer applicable. A more natural approach to model HetNets is by using random spatial models, where the locations of the BSs are assumed to form a realization of a Poisson Point Process (PPP) [1], [2]. This model has the advantages of being scalable to multiple classes and accurate to model location randomness, especially that of the small cells. Additionally, powerful tools from stochastic geometry can be used to derive closed form results for general multi-tier networks, which was not even possible for single tier networks using conventional models [2]. While sufficient progress has been made in modeling single-antenna HetNets [2]–[5], the modeling tools for multi-antenna HetNets have just started to be developed [6].

The main challenges in modeling multi-antenna HetNets lie in handling their complexity, arising primarily due to the number of possible multi-antenna techniques to choose from, not to mention the need for tractable characterization of each technique. In this paper, we develop a general model for K-tier multi-antenna HetNets, where BSs across tiers differ in terms of transmit power, target-SIR, deployment density, number of transmit antennas and multi-antenna technique, such as single user beamforming or multiuser zero-forcing. This can be thought of an extension of the model developed in [2], [7] to the multi-antenna case. Charactering the effect of these techniques by choosing appropriate channel power distributions for the desired and the interfering links, we derive an upper bound on the coverage probability. We then focus on the case where all tiers perform multiuser zero-forcing assuming perfect channel state information (CSI) with the number of users equal to the number of transmit antennas and show that the bound can be reduced to a simple closed form expression, which is tight down to very low target-SIRs. In this case, we also show that the coverage probability is invariant to the density of the BSs, number of tiers and transmit powers when the target-SIRs and the number of antennas are the same for all tiers. Additionally, the coverage probability in multiuser zero-forcing case is always smaller than the single-antenna HetNets due to a limited "proximity" gain as compared to the single-antenna HetNets, whereas the area spectral efficiency is always higher.

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II. System Model

We consider K different classes or tiers of BSs, indexed by the set $\mathcal{K} = \{1, 2, \dots, K\}$. The BSs across tiers differ in terms of their transmit power P_k with which they transmit to each user, deployment density λ_k, target SIR β_k and number of antennas M_k. Each tier of BSs is modeled by an independent Poisson Point Process (PPP) Φ_k with density λ_k. While this assumption is likely more accurate in the case of small cells, it has been validated even for planed tiers, such as macrocells, both by empirical observations [8] and theoretical arguments under sufficient channel randomness [9]. For notational simplicity, we assume that the thermal noise is negligible as compared to the self interference and is hence ignored. This is justified in the current wireless networks, which are typically interference limited [10]. As will be evident from our analysis, thermal noise can be included in the proposed framework with very little extra work.

In this paper, we focus on the downlink analysis performed at a typical single-antenna mobile located at the origin, which is made possible by the Slivnyak's theorem [11]. To develop a general framework in which each class of BS may follow any multi-antenna technique, we assume that the channel power for the direct link from a k^{th} tier BS located at $x \in \mathbb{R}^2$ to the mobile located at origin is denoted by h_{kx} and for the interfering link from a j^{th} tier BS located at $y \in \mathbb{R}^2$ is denoted by g_{jy}. This differentiation is made because the channel power distributions for the direct and interfering links from the same BS are in general different and depend upon the multi-antenna technique adopted by the BS. For a large class of multi-antenna techniques, it can be argued that under Rayleigh fading, the channel power distributions of both the direct and the interfering links follow the Gamma distribution [12]. Therefore, we assume that $h_{kx} \sim \Gamma(\Delta_k, 1)$ and $g_{jy} \sim \Gamma(\Psi_j, 1)$, where Δ_k

and Ψ_j are positive integers that depend upon the number of antennas and the multi-antenna technique adopted by a BS. Some examples of the particular multi-antenna techniques falling in this general framework are [12]:

- $\Delta_k = 1$ and $\Psi_j = M_j$, where M_j is the number of transmit antennas at j^{th} tier. This models the case where multiuser zero-forcing with perfect CSI is employed to serve M_j users at tier j. This case will be henceforth referred to as multi user MIMO (MU-MIMO) case.
- $\Delta_k = M_k$ and $\Psi_j = 1$, $\forall j \in \mathcal{K}$. This models MISO eigen-beamforming or single-user beamforming (SU-BF) with perfect CSI.
- $\Delta_k = \Psi_j = 1, \forall j, k$. This models the case of single-user beamforming with limited feedback (quantized CSIT).

The received power at the typical mobile located at origin from the BS located at $x_k \in \Phi_k$ is:

$$P_r = P_k h_{kx_k} \|x_k\|^{-\alpha}, \tag{1}$$

where α is the path loss exponent. It is important to recall that we assume per user power constraint in this formulation. The received $\mathtt{SIR}$ can now be expressed as:

$$\mathtt{SIR}(x_k) = \frac{P_k h_{kx_k} \|x_k\|^{-\alpha}}{\sum_{k\in\mathcal{K}} \sum_{y\in\Phi_j \setminus x_k} P_j g_{jy} \|y\|^{-\alpha}}. \tag{2}$$

For cell association, we assume that the set of the candidate serving BSs is the collection of the BSs that provide strongest instantaneous received power from each tier. The typical mobile is said to be in coverage if the received $\mathtt{SIR}$ from one of these candidate serving BSs is more than the respective target-$\mathtt{SIR}$, as discussed in detail in the next section.

III. Coverage Probability

This is the main technical section of the paper where we study the coverage probability of a typical mobile user, which is defined as follows:

Definition 1 (Coverage Probability). *A typical mobile user is said to be in coverage if its downlink* $\mathtt{SIR}$ *from at least one of the BSs is higher than the corresponding target. This can be mathematically expressed as:*

$$\mathtt{P_c} = \mathbb{P}\left(\bigcup_{k\in\mathcal{K}} \max_{x_k\in\Phi_k} \mathtt{SIR}(x_k) > \beta_k\right). \tag{3}$$

The coverage probability can be equivalently defined as the average area in coverage or the average fraction of mobile users in coverage. We now derive an upper bound on the coverage probability of a K-tier HetNet, where all the BSs of a particular tier are assumed to adopt the same multi-antenna technique. It is important to note that if a given fraction of BSs of a particular tier independently adopt different multi-antenna techniques, we can divide the original tier into two tiers with appropriate densities, which is enabled by the fact that independently thinning a PPP leads to two independent PPPs. For example, if fraction f of k^{th} tier BSs follow SU-BF and the remaining $(1-f)$ fraction independently follow MU-MIMO, the k^{th} tier can be divided into two tiers modeled as independent PPPs $\Phi_{k^{(1)}}$ and $\Phi_{k^{(2)}}$ with densities $\lambda_{k^{(1)}} = f\lambda_k$ and $\lambda_{k^{(2)}} = (1-f)\lambda_k$, respectively. Before deriving the main result, we first derive an expression for the Laplace transform of interference. The result is given in Lemma 1 and the proof is given in the Appendix.

Lemma 1. *The Laplace transform of interference* $\mathcal{L}_I(s) = \mathbb{E}\left[e^{-sI}\right]$, *where* $I = \sum_{k\in\mathcal{K}} \sum_{y\in\Phi_j} P_j g_{jy} \|y\|^{-\alpha}$ *is*

$$\mathcal{L}_I(s) = \exp\left(-s^{\frac{2}{\alpha}} \sum_{j\in\mathcal{K}} \lambda_j P_j^{\frac{2}{\alpha}} C(\alpha, \Psi_j)\right), \tag{4}$$

where

$$C(\alpha, \Psi_j) = \frac{2\pi}{\alpha} \sum_{m=1}^{\Psi_j} \binom{\Psi_j}{m} B\left(\Psi_j - m + \frac{2}{\alpha}, m - \frac{2}{\alpha}\right), \tag{5}$$

and $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$ *is Euler's Beta function.*

Using this result, the upper bound on the coverage probability can now be derived and the result is given in Theorem 1.

Theorem 1. *The coverage probability of a typical mobile user in* K*-tier HetNet is upper bounded by*

$$\mathtt{P_c} \le \sum_{k\in\mathcal{K}} \lambda_k \sum_{i=0}^{\Delta_k - 1} \frac{1}{i!} \sum \frac{i!}{j_1! j_2! \dots j_i!} \int_{x_k\in\mathbb{R}^2} (-s_{x_k})^i e^{-\mathcal{C} s_{x_k}^{\frac{2}{\alpha}}} \prod_{\ell=1}^{i} \frac{1}{(\ell!)^{jl}} \left(-\mathcal{C} s_{x_k}^{\frac{2}{\alpha}-\ell} \prod_{n=0}^{\ell-1}\left(\frac{2}{\alpha} - n\right)\right)^{j_\ell} \mathrm{d}x_k, \tag{6}$$

where $s_{x_k} = \beta_k \|x_k\|^\alpha P_k^{-1}$, $\mathcal{C} = \sum_{j\in\mathcal{K}} \lambda_j P_j^{\frac{2}{\alpha}} C(\alpha, \Psi_j)$, *and the innermost summation is taken over all* i*-tuples of non negative integers* $(j_1, \dots, j_i)$ *satisfying the constraint* $1\cdot j_1 + 2\cdot j_2 + 3\cdot j_3 + \dots + i\cdot j_i = i$.

Proof: Starting with the definition of the coverage probability:

$$\mathtt{P_c} = \mathbb{E}\left[1\left(\bigcup_{k\in\mathcal{K}} \bigcup_{x_k\in\Phi_k} \mathtt{SIR}(x_k) > \beta_k\right)\right] \tag{7}$$

$$\stackrel{(a)}{\le} \mathbb{E}\left[\sum_{k\in\mathcal{K}} \sum_{x_k\in\Phi_k} 1\left(\mathtt{SIR}(x_k) > \beta_k\right)\right] \tag{8}$$

$$= \sum_{k\in\mathcal{K}} \mathbb{E}\left[\sum_{x_k\in\Phi_k} 1\left(P_k h_{kx_k} \|x_k\|^{-\alpha} > \beta_k I_{x_k}\right)\right], \tag{9}$$

where (a) follows from the union bound and I_{x_k} is the interference received by the typical mobile when it is connected to the k^{th} tier BS located at x_k, i.e.,

$$I_{x_k} = \sum_{j\in\mathcal{K}} \sum_{y\in\Phi_j\setminus x_k} P_j g_{jy} \|y\|^{-\alpha}. \tag{10}$$

Now, since the channel power of the direct link is independent of everything else, we can take the expectation w.r.t. h_{kx_k} inside in (9) to write the coverage probability as:

$$\mathtt{P_c} \le \sum_{k\in\mathcal{K}} \mathbb{E}\left[\sum_{x_k\in\Phi_k} \mathbb{P}\left(h_{kx_k} > \beta_k I_{x_k} \|x_k\|^\alpha P_k^{-1}\right)\right]. \tag{11}$$

Now we first evaluate the probability $\mathbb{P}(h_{kx} > z)$ as follows:

$$\mathbb{P}(h_{kx} > z) \overset{(a)}{=} \frac{\Gamma(\Delta_k, z)}{\Gamma(\Delta_k)} \overset{(b)}{=} e^{-z} \sum_{i=0}^{\Delta_k - 1} \frac{z^i}{i!}, \qquad (12)$$

where (a) follows from $h_{kx} \sim \Gamma(\Delta_k, 1)$, and $\Gamma(\Delta_k, z)$ in the numerator is the upper incomplete Gamma function given by $\Gamma(\Delta_k, z) = \int_z^\infty u^{\Delta_k - 1} e^{-u} \mathrm{d}u$, (b) follows by specializing the expression of incomplete Gamma function for the case when Δ_k is an integer. Now denote $\beta_k \|x_k\|^\alpha P_k^{-1}$ by s_{x_k} and substitute (12) in (11) to get:

$$\mathtt{P_c} \leq \sum_{k \in \mathcal{K}} \mathbb{E} \sum_{x_k \in \Phi_k} e^{-s_{x_k} I_{x_k}} \sum_{i=0}^{\Delta_k - 1} \frac{(s_{x_k} I_{x_k})^i}{i!} \qquad (13)$$

$$\overset{(a)}{=} \sum_{k \in \mathcal{K}} \lambda_k \int_{x_k \in \mathbb{R}^2} \mathbb{E}_{I_{x_k}} e^{-s_{x_k} I_{x_k}} \sum_{i=0}^{\Delta_k - 1} \frac{(s_{x_k} I_{x_k})^i}{i!} \mathrm{d}x_k \qquad (14)$$

$$= \sum_{k \in \mathcal{K}} \lambda_k \sum_{i=0}^{\Delta_k - 1} \frac{1}{i!} \int_{x_k \in \mathbb{R}^2} \mathbb{E}_{I_{x_k}} e^{-s_{x_k} I_{x_k}} (s_{x_k} I_{x_k})^i \mathrm{d}x_k, \qquad (15)$$

where (a) follows from Campbell Mecke Theorem [11]. Now note that if Δ_k were 1, the expectation term is just $\mathcal{L}_{I_{x_k}}(s_{x_k})$, i.e., the Laplace transform of interference evaluated at s_{x_k}. For $\Delta_k > 1$, we evaluate the expectation in terms of the derivative of the Laplace transform as follows:

$$\mathbb{E}_{I_{x_k}} \left[e^{-s_{x_k} I_{x_k}} (s_{x_k} I_{x_k})^i \right] \overset{(a)}{=} s_{x_k}^i \mathcal{L}\{t^i f_{I_{x_k}}(t)\}(s_{x_k}) \qquad (16)$$

$$\overset{(b)}{=} (-s_{x_k})^i \frac{\delta^i}{\delta(s_{x_k})^i} \mathcal{L}_{I_{x_k}}(s_{x_k}), \qquad (17)$$

where (a) follows from the definition of the Laplace transform and (b) follows from the identity $t^n f(t) \longleftrightarrow (-1)^n \frac{\delta^n}{\delta(s)^n} \mathcal{L}\{f(t)\}(s)$. Substituting this in (17), we can express the upper bound on coverage probability in terms of Laplace transform of interference as follows:

$$\mathtt{P_c} \leq \sum_{k \in \mathcal{K}} \lambda_k \sum_{i=0}^{\Delta_k - 1} \frac{1}{i!} \int_{x_k \in \mathbb{R}^2} (-s_{x_k})^i \frac{\delta^i}{\delta(s_{x_k})^i} \mathcal{L}_{I_{x_k}}(s_{x_k}) \mathrm{d}x_k. \qquad (18)$$

Using the Laplace transform expression derived in Lemma 1 and calculating its derivative using Faà di Bruno's formula for a composite function $(f \circ g)(s_{x_k})$, with $f(s_{x_k}) = \exp(s_{x_k})$, and $g(s_{x_k}) = -\mathcal{C} s_{x_k}^{\frac{2}{\alpha}}$, the result follows. ■

We note that the above upper bound is not in closed form but can be easily computed numerically, especially for small values of Δ_k. We now comment on the tightness of this bound.

Remark 1 (Tightness of the bound)**.** *Since the bound is derived by using the union bound in* (8)*, the tightness depends upon the number of candidate BSs that provide* SIR *greater than the target-*SIR*. If there is strictly one such BS, the bound holds with equality, as is the case in single-antenna HetNets for* $\beta_k > 1$ *[2]. In general, the number of such BSs is small and the resulting bound tight when* $\Delta_k < \Psi_k$*, e.g., MU-MIMO, and/or when the target-*SIR*s are high. We will demonstrate this in case of MU-MIMO later in this section.*

In the rest of this section, we will focus on the MU-MIMO case as defined in the previous section. Recall that in this case $\Delta_k = 1$ and $\Psi_j = M_j$, which avoids the step in which we need to derive the derivatives of the Laplace transform leading to a closed form expression for the coverage probability bound. The result is given as the following Corollary and the proof follows directly from the proof of Theorem 1.

Corollary 1. *The coverage probability in a* K*-tier MU-MIMO HetNet with each* k^{th} *tier BS serving* M_k *users is given by*

$$\mathtt{P_c} \leq \pi \frac{\sum_{k \in \mathcal{K}} \lambda_k P_k^{\frac{2}{\alpha}} \beta_k^{-\frac{2}{\alpha}}}{\sum_{j \in \mathcal{K}} \lambda_j P_j^{\frac{2}{\alpha}} C(\alpha, M_j)}. \qquad (19)$$

We will henceforth assume that the closed form upper bound derived in Corollary 1 is tight and can be used as an approximation for the coverage probability in MU-MIMO case. We will validate this assumption in the Numerical Results section. For notational simplicity we will use an equality instead of an approximation.

Remark 2 (Similarity with $\mathtt{P_c}$ in single-antenna case)**.** *The coverage probability expression derived for MU-MIMO case in Corollary 1 has a striking similarity with the coverage probability in the single-antenna case derived in [2]. The only difference is that the constant* $C(\alpha, M_j)$ *in that case is simply* $C(\alpha) = \frac{2\pi^2 \csc\left(\frac{2\pi}{\alpha}\right)}{\alpha}$.

To facilitate direct comparison of the MU-MIMO and the single-antenna cases, let us take a closer look at the expression of $C(\alpha, M)$ given by (5). First note that $C(\alpha, M)$ is an increasing function of M. Now let us evaluate $C(\alpha, 1)$:

$$C(\alpha, 1) = \frac{2\pi}{\alpha} B\left(\frac{2}{\alpha}, 1 - \frac{2}{\alpha}\right) \qquad (20)$$

$$= \frac{2\pi}{\alpha} \Gamma\left(\frac{2}{\alpha}\right) \Gamma\left(1 - \frac{2}{\alpha}\right) = \frac{2\pi^2 \csc\left(\frac{2\pi}{\alpha}\right)}{\alpha}, \qquad (21)$$

where the last step follows by Euler's reflection formula. Hence $C(\alpha, 1)$ is the same as $C(\alpha)$ derived for the single-antenna case in [2]. From the monotonicity of $C(\alpha, M)$ it follows that $C(\alpha, M) > C(\alpha)\ \forall M > 1$.

Remark 3 (MU-MIMO vs single-antenna coverage)**.** *Keeping all the system parameters the same, the MU-MIMO coverage is always lower than that of the single-antenna case.*

Remark 4 (Scale Invariance)**.** *The MU-MIMO coverage probability is invariant to the density of the BSs, number of tiers and the transmit powers when the target-*SIR*s and the number of transmit antennas are the same for all the tiers. The coverage probability in this case is given by* $\mathtt{P_c} = \frac{\pi}{C(\alpha, M)} \beta^{-\frac{2}{\alpha}}$*. This result is again similar to the single-antenna result where the coverage probability reduces to* $\mathtt{P_c} = \frac{\pi}{C(\alpha)} \beta^{-\frac{2}{\alpha}}$.

Although the comparison of MU-MIMO and single-antenna cases in terms of coverage probabilities is quite conclusive, it does not account for the fact that MU-MIMO is serving a higher density of users (assuming the density of the BSs is the same in both the cases). To account for this fact, we consider area spectral efficiency (ASE), which gives the number of bits transmitted per unit area per unit time per unit bandwidth. For

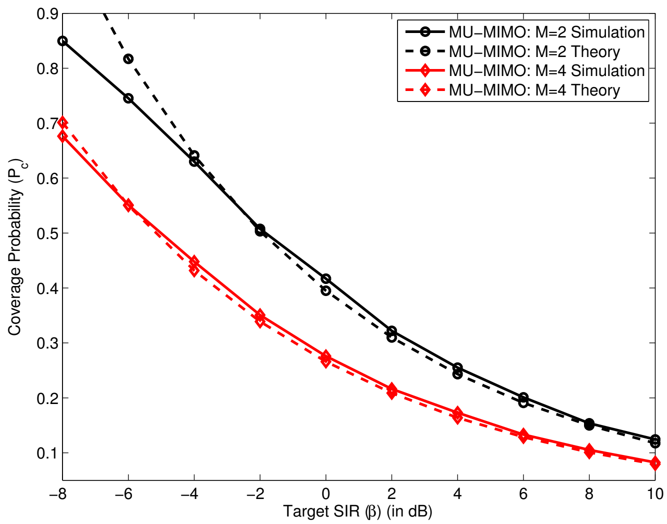

Fig. 1. Coverage probability of a two tier HetNet when both the tiers perform MU-MIMO ($K = 2, P = [1, .01], M_1 = M_2 = M, \lambda_2 = 2\lambda_1, \beta_1 = \beta_2, \alpha = 3.8$).

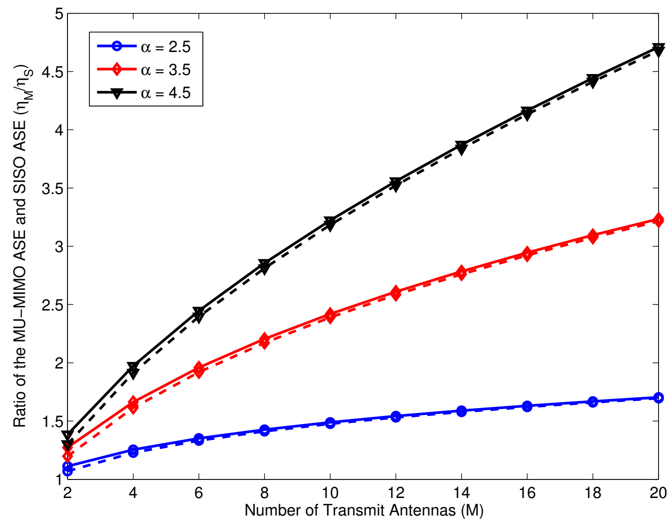

Fig. 2. The ratio of the MU-MIMO ASE and the single-antenna ASE given by (24). The dotted plots correspond to the approximation of the ratio given by (26).

simplicity of exposition, we will focus on the case where the coverage probabilities in both the MU-MIMO and the single-antenna cases are scale invariant. The ASE for the MU-MIMO case is:

$$\eta_M = M \frac{\pi}{C(\alpha, M)} \beta^{-\frac{2}{\alpha}} \log_2(1+\beta) \sum_{k \in K} \lambda_k, \tag{22}$$

and for the single-antenna case is:

$$\eta_S = \frac{\pi}{C(\alpha)} \beta^{-\frac{2}{\alpha}} \log_2(1+\beta) \sum_{k \in K} \lambda_k. \tag{23}$$

The ratio of the ASEs can be expressed as:

$$\frac{\eta_M}{\eta_S} = \frac{M C(\alpha)}{C(\alpha, M)}. \tag{24}$$

Using the fact that

$$\lim_{M \to \infty} \frac{C(\alpha, M)}{M^{\frac{2}{\alpha}}} = \pi \Gamma(1 - 2/\alpha), \tag{25}$$

the ratio of the ASEs can be approximated as:

$$\frac{\eta_M}{\eta_S} \approx \frac{M^{1-\frac{2}{\alpha}} C(\alpha)}{\pi \Gamma(1-2/\alpha)} \tag{26}$$

$$= \Gamma\left(1 + \frac{2}{\alpha}\right) M^{1-\frac{2}{\alpha}}, \tag{27}$$

which shows that the ratio grows with the number of antennas when $\alpha > 2$. In the next section we will validate this observation and show that the ASE in case of MU-MIMO case is always higher than the single-antenna case. We will also show that the approximation is surprisingly tight even for small values of M.

IV. Numerical Results

Since there is a slight difference in the simulation of the proposed multi-antenna model and the ones proposed in the literature for the single-antenna HetNets, e.g., [2], we will briefly summarize the simulation procedure before explaining the results. Choose a sufficiently large window and simulate the locations of different classes of BSs as realizations of independent PPPs of given densities. Associate two independent marks h_x and g_x with each BS. Assuming the typical mobile lies at the origin, calculate the desired signal strength from each BS using the sequence of marks $\{h_{kx}\}$ and the the interference power using the sequence $\{g_{kx}\}$. Calculate the received SIR from each BS. The mobile is now said to be in coverage if the received SIR from at least one of the BSs is more than the corresponding target. Repeating this procedure sufficient number of times, we have an estimate of the coverage probability.

Using this procedure, we now evaluate the coverage probability of a two tier HetNet in MU-MIMO case and compare the results with the upper bound derived in Corollary 1 in Fig. 1. As stated in Remark 1, the bound is tight down to very low target SIRs. Even for $M = 2$, the bound is tight down to about -4 dB. A slight gap at moderate to high target-SIRs is due to the edge effects in simulation, also observed earlier in [2]. This validates our assumption of considering the upper bound as an approximation of the coverage probability in case of MU-MIMO in the previous section.

We next plot the ratio of the ASEs of the MU-MIMO and single-antenna cases given by (24) in Fig. 2. As stated in the previous section, the ASE of MU-MIMO case is always higher and the gap increases further at higher path loss exponents or when the number of transmit antennas increases. We also note that the approximation given by (26) is extremely tight even when the number of antennas is small.

We now additionally consider single user beamforming, the case corresponding to $\Delta_k = 1, \Psi_j = M_j$, along with MU-MIMO and single-antenna cases in Fig. 3. We study the effect of adding a second tier to the network, where both the first and the second tier can be one of the three possible types: i) single-antenna (SISO), ii) MU-MIMO, iii) SU-beamforming. This result shows that the case where both the tiers perform

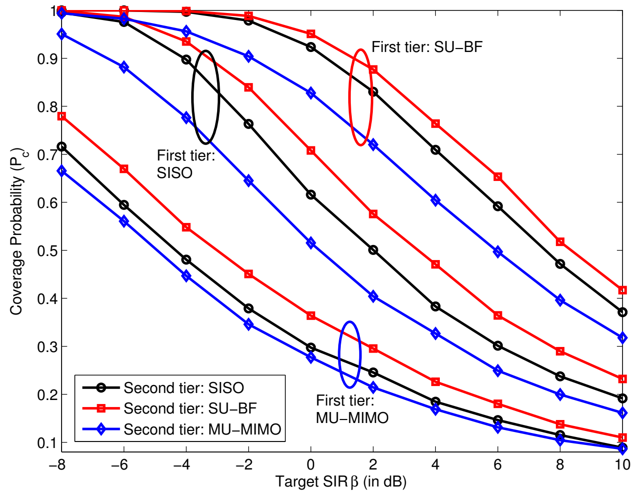

Fig. 3. Comparison of the coverage probability in a two tier HetNet for various combinations of multi-antenna techniques ($K = 2, P = [1, .01], \lambda_2 = 2\lambda_1, \beta_1 = \beta_2, \alpha = 3.8$). The number of antennas in case of multi-antenna tiers is $M = 4$.

SU-beamforming results in the highest coverage, whereas the case where both the tiers perform MU-MIMO leads to the lowest coverage. This is intuitive because SU-beamforming case has an additional beamforming gain; in addition to the proximity gain enjoyed by the single-antenna case.

V. Conclusions

In this paper, we developed a tractable model to study coverage in multi-antenna HetNets. For a general model where each class of BSs may adopt a different multi-antenna technique, we derive an upper bound on the coverage probability. We show that the bound can be reduced to a useful closed form expression in MU-MIMO case. Using this result, we show that MU-MIMO leads to a lower coverage but higher ASE as compare to single-antenna HetNets for the same deployment density. We also numerically show that the SU-beamforming case leads to the highest coverage due to the additional beamforming gain. The general extensions of this work include carefully comparing coverage, ASE and average rate in more general deployment scenarios. Specific to the analysis presented in this paper, it is important to specialize the coverage probability bound for SU-beamforming and study its tightness, which will facilitate its ASE comparison with single-antenna and MU-MIMO cases.

Appendix: Proof of Lemma 1

The Laplace transform of interference $\mathcal{L}_I(s) = \mathbb{E}_{I_{x_k}}\left[e^{-sI}\right]$ can be derived as follows:

$$\mathbb{E}_I\left[e^{-sI}\right] = \mathbb{E}_I\left[e^{-s\sum_{j\in\mathcal{K}}\sum_{y\in\Phi_j} P_j g_{jy}\|y\|^{-\alpha}}\right] \quad (28)$$

$$\stackrel{(a)}{=} \prod_{j\in\mathcal{K}} \mathbb{E}\left[\prod_{y\in\Phi_j} e^{-sP_j g_{jy}\|y\|^{-\alpha}}\right] \quad (29)$$

$$\stackrel{(b)}{=} \prod_{j\in\mathcal{K}} \mathbb{E}_{\Phi_j}\left[\prod_{y\in\Phi_j} \mathcal{L}_{g_{jy}}\left(sP_j\|y\|^{-\alpha}\right)\right] \quad (30)$$

$$\stackrel{(c)}{=} \prod_{j\in\mathcal{K}} \exp\left(-\lambda_j\int_{\mathbb{R}^2}\left(1-\mathcal{L}_{g_{jy}}\left(sP_j\|y\|^{-\alpha}\right)\right)\mathrm{d}y\right) \quad (31)$$

$$\stackrel{(d)}{=} \prod_{j\in\mathcal{K}} \exp\left(-\lambda_j\int_{\mathbb{R}^2}\left(1-\frac{1}{(1+sP_j\|y\|^{-\alpha})^{\Psi_j}}\right)\mathrm{d}y\right) \quad (32)$$

$$= \prod_{j\in\mathcal{K}} \exp\left(-\lambda_j\int_{\mathbb{R}^2}\frac{(1+sP_j\|y\|^{-\alpha})^{\Psi_j}-1}{(1+sP_j\|y\|^{-\alpha})^{\Psi_j}}\mathrm{d}y\right) \quad (33)$$

$$\stackrel{(e)}{=} \prod_{j\in\mathcal{K}} \exp\left(-\lambda_j\int_{\mathbb{R}^2}\frac{\sum_{m=1}^{\Psi_j}\binom{\Psi_j}{m}(sP_j\|y\|^{-\alpha})^m}{(1+sP_j\|y\|^{-\alpha})^{\Psi_j}}\mathrm{d}y\right) \quad (34)$$

$$= \prod_{j\in\mathcal{K}} \exp\left(-\lambda_j\sum_{m=1}^{\Psi_j}\binom{\Psi_j}{m}\int_{\mathbb{R}^2}\frac{(s_{x_k}P_j\|y\|^{-\alpha})^m}{(1+s_{x_k}P_j\|y\|^{-\alpha})^{\Psi_j}}\mathrm{d}y\right) \quad (35)$$

$$\stackrel{(f)}{=} \exp\left(-s_{x_k}^{\frac{2}{\alpha}}\sum_{j\in\mathcal{K}}\lambda_j P_j^{\frac{2}{\alpha}} C(\alpha,\Psi_j)\right), \quad (36)$$

where (a) follows from the independence of the tiers, (b) follows from the fact that channel powers are independent of the BS locations, (c) follows from PGFL of PPP [11], (d) follows from the Laplace transform of the $g_{jy} \sim \Gamma(\Psi_j, 1)$, (e) follows from Binomial theorem, and (f) follows from converting to cartesian to polar coordinates followed by substituting $(1+r^{-\alpha})^{-1} \to t$ to convert the integral into Euler's Beta function $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$.

References

[1] J. G. Andrews, F. Baccelli, and R. K. Ganti, "A tractable approach to coverage and rate in cellular networks," *IEEE Trans. on Communications*, vol. 59, no. 11, pp. 3122–3134, Nov. 2011.

[2] H. S. Dhillon, R. K. Ganti, F. Baccelli, and J. G. Andrews, "Modeling and analysis of K-tier downlink heterogeneous cellular networks," *IEEE Journal on Sel. Areas in Communications*, vol. 30, no. 3, pp. 550 – 560, Apr. 2012.

[3] H.-S. Jo, Y. J. Sang, P. Xia, and J. G. Andrews, "Heterogeneous cellular networks with flexible cell association: A comprehensive downlink SINR analysis," *IEEE Trans. on Wireless Communications*, vol. 11, no. 10, pp. 3484 – 3495, Oct. 2012.

[4] S. Mukherjee, "Distribution of downlink SINR in heterogeneous cellular networks," *IEEE Journal on Sel. Areas in Communications*, vol. 30, no. 3, pp. 575 – 585, Apr. 2012.

[5] H. S. Dhillon, R. K. Ganti, F. Baccelli, and J. G. Andrews, "Coverage and ergodic rate in K-tier downlink heterogeneous cellular networks," in *Proc., Allerton Conf. on Comm., Control, and Computing*, Monticello, IL, Sep. 2011.

[6] R. W. Heath, Jr. and M. Kountouris, "Modeling heterogeneous network interference," in *Proc., Information Theory and its Applications (ITA)*, San Deigo, CA, Feb. 2012.

[7] H. S. Dhillon, R. K. Ganti, and J. G. Andrews, "A tractable framework for coverage and outage in heterogeneous cellular networks," in *Proc., Information Theory and its Applications (ITA)*, San Diego, CA, Feb. 2011.

[8] D. B. Taylor, H. S. Dhillon, T. D. Novlan, and J. G. Andrews, "Pairwise interaction processes for modeling cellular network topology," in *Proc., IEEE Globecom*, Anaheim, CA, Dec. 2012.

[9] B. Blaszczyszyn, M. K. Karray, and H.-P. Keeler, "Using poisson processes to model lattice cellular networks," available online: arxiv.org/abs/1207.7208.

[10] G. Boudreau, J. Panicker, N. Guo, R. Chang, N. Wang, and S. Vrzic, "Interference coordination and cancellation for 4G networks," *IEEE Communications Magazine*, vol. 47, no. 4, pp. 74 – 81, Apr. 2009.

[11] D. Stoyan, W. S. Kendall, and J. Mecke, *Stochastic Geometry and Its Applications*, 2nd ed. Chichester: John Wiley and Sons, 1995.

[12] H. Huang, C. B. Papadias, and S. Venkatesan, *MIMO Communication for Cellular Networks*. Springer, 2012.