Localization Performance in Cellular Networks

Javier Schloemann, Harpreet S. Dhillon, and R. Michael Buehrer

Abstract—When the Global Positioning System is unavailable, cellular networks become the dominant vehicle for positioning. However, no tractable approach exists for gaining general insights into localization performance in such networks. Instead, analysis is often done using deterministic network models or with complex system-level simulations, resulting in highly context-specific insights, which do not translate well to random network topologies. In this paper, we motivate and introduce a new approach for analyzing localization performance in cellular networks using tools from point process theory and stochastic geometry. After presenting the model, easy-to-use expressions are derived for the distributions of base station hearability, a metric which is closely-related to localization performance, with and without base station coordination.

Index Terms—Localization, cellular networks, hearability, stochastic geometry, Poisson point processes.

I. INTRODUCTION

THE STUDY OF LOCALIZATION in cellular networks has been around for many years [1], [2]. The driving force behind much of the research is a mandate by the Federal Communications Commission requiring cellular network operators to locate those calling 911 to within certain accuracy requirements [3]. Until recently, these requirements included only outdoor location accuracies. Accordingly, the predominant way cellular network operators have met the requirements of the mandate is by relying on the Global Positioning System (GPS). In January of 2015, the FCC expanded its mandate to include a phase-in of indoor positioning requirements, citing that the bulk of emergency calls now originate indoors [4]. While GPS is reliably available under clear sky conditions, where it provides accurate outdoor positioning, its availability suffers greatly in both *urban canyons* and indoor scenarios. In these situations, the dominant vehicle for positioning is a fallback to cellular network localization. Presently, however, no approaches exist to analyze the fundamentals of localization performance in such networks. In this paper, we present a new analytical model exploiting concepts from point process theory [5] and stochastic geometry [6] that lends tractability to the study of cellular network localization. Using this model, we derive easy-to-use expressions which provide insight into localization performance with and without base station (BS) coordination.

A. Prior Art and Motivation

In large part, the lack of a tractable analytical model is due to the fact that BS locations are the result of a very complicated optimization problem which is difficult to model and depends

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upon various system parameters. Broadly speaking, BSs are deployed in locations so as to maximize coverage and rate for communication purposes, yet the actual placements are further subject to topographical limitations. Moreover, the BS deployments which optimize communication goals may not be sympathetic to localization goals, potentially resulting in highly suboptimal geolocation systems. Consider that in communication systems, it is ideal for a mobile device to receive a strong signal from its serving BS and weak signals from all its neighbors, whereas localization is not possible unless a mobile device receives usable signals from these neighbors, with accuracy generally improving as more neighbors are heard. To underscore the conflicting nature of the two goals, the need to cater to localization demands has a name among cellular system designers, who refer to it as the *hearability problem*, due to the fact that increasing hearability from neighboring cells for the purposes of positioning is contrary to the principles of cellular system design [7]. Despite this, the localization literature often presents results using specific device and network layouts, which are typically favorable to location estimation (such as by forcing the device to be localized inside the convex hull of its anchors), but do not necessarily represent what may generally be encountered in cellular networks. Even when the device location is not fixed, but the network is, such as when modeling cellular networks using the popular hexagonal grid model [8], analytical results are difficult to come by. Instead, network designers resort to complex system-level simulations, as is clear from the 3GPP standardization process [9]. Unfortunately, simulation approaches do not typically allow general insights to be obtained, such as how localization is impacted by network design parameters. This motivates the need for tractable analytical models that will provide preliminary design insights and will either circumvent the need for simulations completely or limit the ranges of the simulation parameters.

B. Contributions

In order to provide a tractable analytical model, we turn to point process theory and stochastic geometry to study the geolocation problem. These tools have been applied in wireless contexts before [10]. In fact, both cellular and ad-hoc networks have been studied using tools from stochastic geometry, with ad-hoc networks being the subject of most of the classical works. Recently, however, tools from stochastic geometry have been increasingly applied to the study of cellular networks [11] and emerging heterogeneous networks [12]. Motivated by these advances, we present a new approach catered to the operation of localization systems and provide analytical expressions for BS hearability with and without BS coordination.

II. SYSTEM MODEL

A. Spatial base station layout

The locations of the BSs are modeled using a homogeneous PPP $\Phi \in \mathbb{R}^2$ with density λ [6]. Due to the stationarity of a homogeneous PPP, the device to be localized is assumed to be located at the origin o. If the interference is treated as noise at the receiver, the most appropriate metric that captures link quality is the signal-to-interference-plus-noise ratio (SINR). For the link from some BS $x \in \Phi$ to the origin, the SINR can be expressed as:

$$\mathsf{SINR}_x = \frac{P\mathcal{S}_x\|x\|^{-\alpha}}{\sum_{\substack{y \in \Phi \\ y \neq x}} P\mathcal{S}_y\|y\|^{-\alpha} + \sigma^2}, \qquad (1)$$

where P is the transmit power, $\mathcal{S}_z$ denotes the independent shadowing affecting the signal from BS z to the origin, $\alpha > 2$ is the pathloss exponent, and σ^2 is the noise variance. Note that (1) represents the SINR prior to any processing gain, yet as will be evident in the sequel, positioning systems typically have to work at lower target SINRs, thereby necessitating the need for some form of processing gain. In general, the post-processing SINR will include some multiplicative factor γ representing the processing gain, which depends upon system parameters (e.g. integration time) and is assumed to average out the effect of small scale fading. This is the reason why the SINR expression in (1) does not contain a fast fading term, which is consistent with current models for evaluating cellular positioning performance [8].

B. Selection of participating base stations

As is common in communication system analyses, serving BSs are selected according to the *strongest BS association policy*, measured using average signal strength, which typically includes long time-scale effects such as shadowing and pathloss. Now, consider that we desire to use a total of L BSs for positioning. Thus, we assume that the L BSs which provide the highest average received power make up the set of *participating BSs*. Their successful participation, however, is not guaranteed as there is some post-processing SINR threshold β (or equivalently, pre-processing SINR threshold β/γ) above which the signals from the participating BSs must arrive in order for them to successfully contribute to the localization procedure. In the absence of shadowing, the set of potential BSs simply corresponds to the set of the L nearest BSs. When shadowing is considered and BSs are selected according to average signal strength, the effect of shadowing may be absorbed as a perturbation in the locations of the BSs provided that the fractional moment $\mathbb{E}\left[\mathcal{S}_z^{2/\alpha}\right] < \infty$ [13], [14]. Thus, when this condition is fulfilled and without loss of generality, we define a new equivalent PPP with density $\lambda \mathbb{E}\left[\mathcal{S}_z^{2/\alpha}\right]$. In doing so, we ensure that the strongest BS association policy in the original PPP is equivalent to the *nearest BS association policy* in the transformed PPP. For notational simplicity, we will continue to represent the transformed PPP by Φ with density λ, under the assumption that if shadowing is present, it
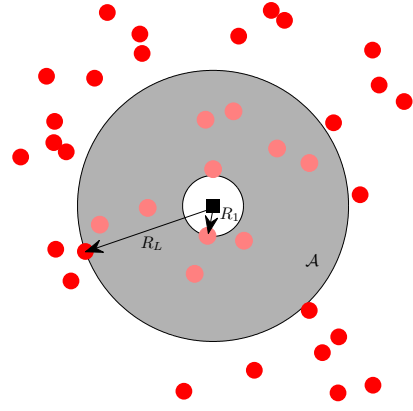

Fig. 1. THE PROPOSED MODEL. The mobile device is denoted by the central square and BSs are denoted by the circles. The BSs participating in the localization procedure (within the gray annular region $\mathcal{A}$) are either all on or all off during the Lth BS's transmission, depending on the coordination.

is already reflected in the density of Φ. Note that the condition on the fractional moment is fairly mild and will almost always be satisfied by the distributions of interest, including the most common assumption of log-normal shadowing with finite mean and standard deviation [14]. As a final note, it has been shown that for SINR, shadowing causes even more regular network models, such as the common hexagonal lattice, to behave like a PPP model [15], [16]. This further validates the use of a PPP to model BS locations.

C. Base station coordination

In cellular networks, backhaul connections allow for the coordination of transmissions between BSs during localization procedures. An example of this technique for improving the hearability of far away BSs is that of *joint scheduling* in LTE [7]. For simplicity of exposition, let us order the BSs in the now shadowing-transformed Φ in terms of increasing distance from the origin such that the location of the kth farthest BS from the origin is denoted by $x_k \in \Phi$. We now enrich the previous SINR expression in (1) for the signals arriving from the participating BSs (i.e., x_k for $k \in \{1, \ldots, L\}$) by including BS coordination:

$$\mathsf{SINR}_k(L) = \frac{P\|x_k\|^{-\alpha}}{\mathbb{1}_{\text{NC}} \cdot \sum_{\substack{i=1 \\ i \neq k}}^{L} P\|x_i\|^{-\alpha} + \sum_{j=L+1}^{\infty} P\|x_j\|^{-\alpha} + \sigma^2}, \qquad (2)$$

where $\mathbb{1}_{\text{NC}}$ is an indicator taking value unity when there is no coordination between BSs. Note that SINR_k is a function of L because of the potential coordination among the L participating BSs.

D. Base station range distributions

From (2), it is clear the SINRs are dependent on the distances of the BSs from the origin, rather than their exact locations. Thus, it is worthwhile to characterize these. Denote the distance of the kth BS from the origin by $R_k = \|x_k\|$ as illustrated in

Figure 1 for the closest and Lth closest BSs. The distribution of R_L is known to be [17]:

$$f_{R_L}(r) = e^{-\lambda \pi r^2} \frac{2(\lambda \pi r^2)^L}{r(L-1)!}. \quad (3)$$

Now, conditioned on the location (or distance) of the Lth BS, we present a useful lemma for understanding the distribution of the $L-1$ closest BSs.

Lemma 1. *Conditioned on the location of the Lth BS, the $L-1$ BSs closer to the origin are distributed according to a binomial point process (BPP) (i.e., in a uniformly random manner) inside the circle of radius R_L centered at the origin.*

Proof. See Appendix A. ∎

Using Lemma 1, we arrive at the following distribution for the most dominant (i.e., closest) interferer, given the distance of the Lth BS. This is formally presented below.

Lemma 2. *The cumulative distribution function of the closest BS distance R_1 given R_L is*

$$F_{R_1|R_L}(r|R_L) = 1 - \left(\frac{R_L^2 - r^2}{R_L^2} \right)^{L-1}. \quad (4)$$

Proof. See Appendix B. ∎

Let $\mathcal{A} = \mathbb{b}(o, R_L) \backslash \mathbb{b}(o, R_1)$, where $\mathbb{b}(\boldsymbol{\theta}, r)$ represents a ball of radius r centered at $\boldsymbol{\theta}$. We have the following lemma characterizing the distribution of the $L-2$ remaining participating BSs in this annular region shown in Figure 1.

Lemma 3. *Conditioned on R_1 and R_L, the $L-2$ BSs located inside the annular region $\mathbb{b}(o, R_L) \backslash \mathbb{b}(o, R_1)$ are distributed according to a BPP.*

Proof. See Appendix A. ∎

III. LOCALIZATION PERFORMANCE

Regardless of the technique, cellular geolocation performance fundamentally depends upon three things: *(i) the geometry of the participating BSs relative to the device being localized, (ii) the number of participating BSs, and (iii) the accuracy of the positioning observations.* These three factors are all highly interdependent. For instance, both the number of BSs whose signals arrive with sufficiently-high SINRs to participate in the localization procedure and the quality of their positioning observations are strongly dependent on the interference field, which is itself driven by the realized network geometry. Thus, a full characterization of localization performance would take into account all possible *geometric conditionings* of the BSs, a task which is extremely difficult. However, by taking into account even just the number of participating BSs, valuable insights into localization performance may be gleaned. In the present work, we focus on the case of $\alpha = 4$ in interference-limited networks, which is commonly of interest in the cellular communication literature. Note that using $\alpha = 4$ allows us to obtain simplified analytical expressions, while remaining close to the pathloss exponent of 3.76 accepted for 3GPP simulations [8]. For more general pathloss exponents, the reader is referred to [18].
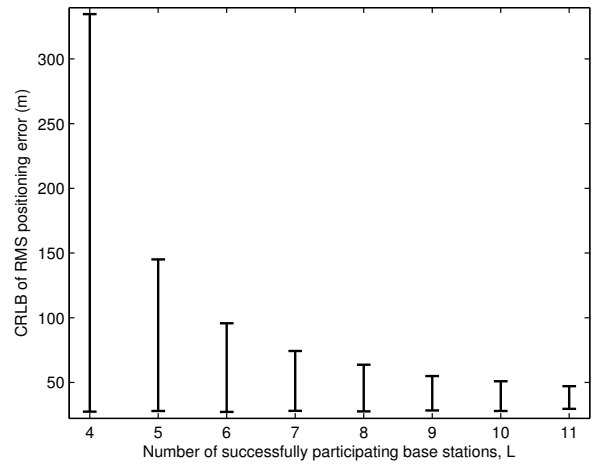

Fig. 2. RELATIONSHIP BETWEEN LOCALIZATION ACCURACY AND BS PARTICIPATION: Ranges of the lower 95$^{\text{th}}$ percentiles of positioning accuracies using TDOA positioning. (No BS coordination, $\alpha = 4$, $\beta/\gamma = -16$ dB, and 8 dB shadowing standard deviation.)
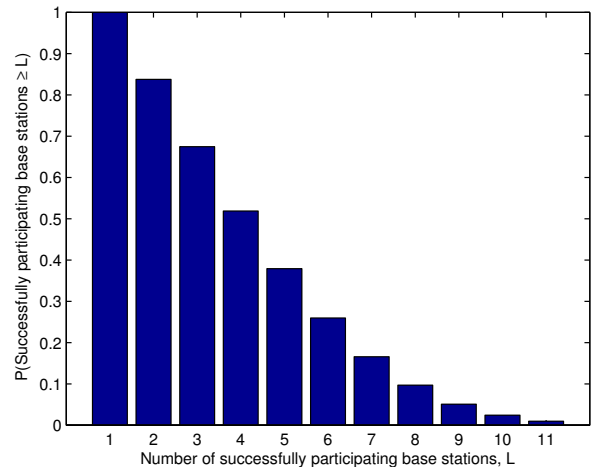

Fig. 3. DECREASING AVAILABILITY OF BSS: A histogram illustrating the decreasing probability of attaining an increasing number of successful BS connections. (The setup is identical to that of Figure 2.)

A. Base station participation and geolocation performance

A clear relationship exists between the number of BSs involved in positioning and the distribution of the resulting localization accuracy. In order to see this, consider the Cramér-Rao lower bound (CRLB), which provides a lower bound on the achievable performance of an unbiased estimator, applied to time-difference-of-arrival (TDOA) positioning, commonly employed in cellular localization (e.g., observed time-difference-of-arrival, OTDOA, in LTE). Using the model described in the previous section with a BS deployment density equivalent to that of an infinite hexagonal grid with 500m intersite distances and a shadowing standard deviation of 8 dB, a localization system without BS coordination was simulated requiring $\beta = -6$ dB for successful signal detection after a processing gain of $\gamma = 10$ dB. The threshold β determines which BSs may successfully participate in the localization procedure, while the BSs' exact SINRs determine the accuracies

of their individual ranging observations, calculated using the well-known TOA ranging CRLB [19] with an assumed signal bandwidth of 2 MHz. Lastly, as is typical, the strongest BS is selected as the reference BS for TDOA. From Figure 2, which plots the range of the lower 95th percentiles of the achievable root-mean-square (RMS) positioning errors observed in the simulation versus the number of successfully participating BSs, we see that not only does the achievable localization performance generally improve with an increasing number of BSs, it starts to become almost predictable. Motivated by this trend, we now formally propose a metric to study the number of BSs able to successfully participate in a localization procedure for a given target SINR β and processing gain γ.

Definition 1 (Participation metric). *For a given BS deployment $\phi \in \Phi$, let Υ represent the maximum number of BSs capable of successfully participating in a localization procedure. This can be mathematically defined as:*

$$\Upsilon = \arg \max_\ell \ \ell \cdot \prod_{k=1}^{\ell} \mathbb{1} \left(\mathrm{SINR}_k(\ell) \geq \frac{\beta}{\gamma} \right). \qquad (5)$$

Note that this metric differs from a traditional metric such as the CRLB in that it is not directly tied to a specific positioning accuracy value since it does not take into account geometry, though the above analysis shows that it is certainly related to performance. Its advantage lies in its tractability for random geometries, whereas the CRLB is easily derived for *deterministic geometries*, but quickly becomes intractable to characterize for random network geometries.

B. Participation from a desired number of base stations

Figure 3 plots the probabilities of attaining some desired number of successful BS connections as observed through the simulation. While generally better localization performance can be attained by increasing BS participation, it is clear from Figure 3 that the probability of successfully involving more BSs decreases sharply as the number of BSs increases. Thus, it is desirable to understand exactly how the participation metric Υ is impacted by network design parameters.

Definition 2 (*L*-localizability probability). *For a given $\phi \in \Phi$, a mobile device is said to be L-localizable if at least L BSs may successfully participate in the localization procedure. The probability of this occurring is simply:*

$$\mathrm{P_L} = \mathbb{P} \left(\Upsilon \geq L \right). \qquad (6)$$

Note that for $L = 1$, this simply gives the downlink coverage probability.

When the small scale fading is averaged out, as is the case in the proposed model described in Section II, it is straightforward to infer from (2) that $\mathbb{1}\left(\mathrm{SINR}_k(L) \geq \beta\right) \geq \mathbb{1}\left(\mathrm{SINR}_l(L) \geq \beta\right)$ for all $k \leq l \leq L$. This simply means that the received SINR from a BS farther from the mobile device is lower than that of a closer BS, meaning that the probability of *L*-localizability can be equivalently expressed as

$$\mathrm{P_L} = \mathbb{E}_\Phi \left[\mathbb{1} \left(\mathrm{SINR}_L(L) \geq \beta \right) \right]. \qquad (7)$$

C. Probability of *L*-localizability

For the signal arriving from the Lth participating BS ($L > 1$), let $\mathcal{I}_1 = \sum_{i=2}^{L-1} P\|x_i\|^{-\alpha}$ be the (potential) interference from the other participating BSs excluding the closest BS (for reasons which will be evident later) and $\mathcal{I}_2 = \sum_{j=L+1}^{\infty} P\|x_j\|^{-\alpha}$ be the interference from BSs beyond the Lth one. We now consider an approximation of $\mathrm{P_L}$ when $\mathbb{1}_{\mathrm{NC}} = 1$, i.e., without BS coordination, by considering the means of $\mathcal{I}_1$ and $\mathcal{I}_2$ conditioned on R_1 and R_L, rather than $\mathcal{I}_1$ and $\mathcal{I}_2$ themselves. These are characterized in the following lemmas.

Lemma 4. *The expected value of $\mathcal{I}_1$ conditioned on R_1 and R_L is*

$$\mathbb{E}[\mathcal{I}_1 | R_1, R_L] = \frac{2P(L-2)}{2 - \alpha} \cdot \frac{R_L^{2-\alpha} - R_1^{2-\alpha}}{R_L^2 - R_1^2}. \qquad (8)$$

Proof. See Appendix C. ∎

Lemma 5. *The expected value of $\mathcal{I}_2$ conditioned on R_L is*

$$\mathbb{E}[\mathcal{I}_2 | R_L] = \frac{2P\pi\lambda}{\alpha - 2} R_L^{2-\alpha}, \qquad (9)$$

for $\alpha > 2$, and unbounded otherwise.

Proof. See Appendix D. ∎

Since cellular networks are typically limited by interference rather than thermal noise, we remove the thermal noise term, and apply Lemmas 4 and 5 to obtain the following approximation for $\mathrm{P_L}$.

Theorem 1 (No base station coordination). *When all BSs transmit simultaneously, the probability of L-localizability may be approximated as*

$$\mathrm{P_L}(L, \beta, \gamma, \lambda) = \int_0^{\sqrt{\frac{\gamma/\beta - (L-1)}{\pi\lambda}}} \left(1 - \frac{1}{\sqrt{\gamma/\beta - \pi\lambda r^2 + \frac{(L-2)^2}{4} - \frac{L-2}{2}}} \right)^{L-1} e^{-\lambda\pi r^2} \frac{2(\lambda\pi r^2)^L}{r(L-1)!} \mathrm{d}r. \qquad (10)$$

Proof. See Appendix E. ∎

Next, we consider the cellular localization scenario when $\mathbb{1}_{\mathrm{NC}} = 0$, i.e., with perfect BS coordination, by again considering the conditional mean of $\mathcal{I}_2$. In this case, Lemma 5 may be employed to obtain a simple approximation of $\mathrm{P_L}$.

Theorem 2 (Perfect base station coordination). *When all BSs transmit sequentially and remain quiet during the other BSs' transmissions, $\mathrm{P_L}$ may be approximated as*

$$\mathrm{P_L}(L, \beta, \gamma, \lambda) = 1 - \sum_{\ell=0}^{L-1} e^{-\frac{\gamma}{\beta}} \frac{\left(\frac{\gamma}{\beta}\right)^\ell}{\ell!}. \qquad (11)$$

Proof. Following a similar derivation to (10), but excluding the interference from the other $L - 1$ participating BSs,

$$\mathrm{P_L}(L, \beta, \gamma, \lambda) = \int_0^{\sqrt{\frac{\gamma/\beta}{\pi\lambda}}} e^{-\lambda\pi r^2} \frac{2(\lambda\pi r^2)^L}{r(L-1)!} \mathrm{d}r, \qquad (12)$$
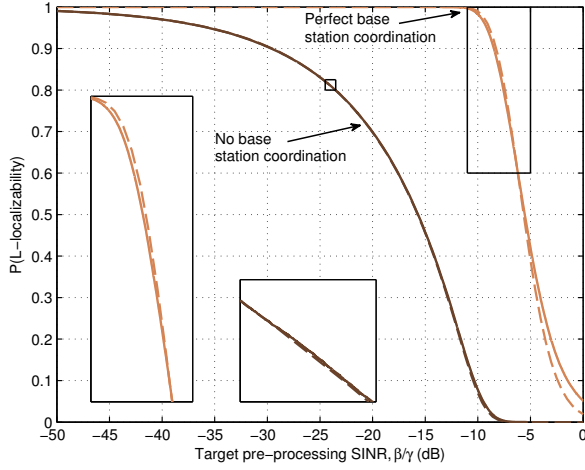
Fig. 4. ACCURACY OF THE APPROXIMATION: This figure illustrates both the true localizability probabilities (solid line) as well as those provided by the approximation (dashed line) for L=4. Note that the approximation is nearly indistinguishable from truth and requires only a simple analytical evaluation.

which is simply the probability of finding *at least* L nodes inside $\mathtt{b}\left(o, \sqrt{\frac{\gamma/\beta}{\pi\lambda}}\right)$. ∎

One interesting remark here is that with perfect BS coordination, the density of the BS deployment does not appear to affect $\mathrm{P_L}$.

IV. NUMERICAL VERIFICATION

We now verify the accuracy of the approximations by comparing with the true $\mathrm{P_L}$ gathered using 10,000 random network realizations. In Figure 4, both approximations are plotted for $L = 4$, the minimum number of BSs required for unambiguous TDOA positioning. We observe that the approximations are remarkably tight, being nearly indistinguishable from truth except in the low-reliability regime ($\mathrm{P_L} < 0.4$), which is not very desirable for system design anyway. A key factor enabling these tight approximations is that the locations of the strongest BS (i.e., the dominant interferer) and the Lth BS are considered *exactly*. To appreciate the value of the tight approximations, consider the exact expression for the k-coverage problem in the absence of fast fading in [20, Corr. 7], which applies here to $\mathrm{P_L}$ with no BS coordination. A cursory glance at the expression reveals its cumbersome nature, involving many multi-fold integrals, from which it is extremely difficult to gain general insights.

V. CONCLUSION

We have introduced concepts from point process theory and stochastic geometry to increase the tractability of localization performance analyses in random networks, which may encompass cellular networks as well as WiFi access point deployments. This is in contrast to most previous approaches, which are either specific to deterministic deployments or rely on time-consuming simulations. Furthermore, the value of the model was demonstrated through tight approximations of

network hearability which make evident the effects of individual network parameters on localization performance.

APPENDIX

A. Proof of Lemmas 1 and 3

Let $i < j$, $n \le \Xi$, Ξ be the number of BSs between x_i and x_j, $A = \mathtt{b}(o, \|x_i\| + da/2)\backslash\mathtt{b}(o, \|x_i\| - da/2)$, $B = \mathtt{b}(o, \|x_j\| + db/2)\backslash\mathtt{b}(o, \|x_j\| - db/2)$, $C = \mathtt{b}(o, \|x_j\| - db/2)\backslash\mathtt{b}(o, \|x_i\| + da/2)$, $D \subseteq C$, $\mathtt{b}(\boldsymbol{\theta}, r)$ represent a ball of radius r centered at $\boldsymbol{\theta}$, and $N_{\mathcal{H}}$ be the number of points in region $\mathcal{H}$. Then,

$$\mathbb{P}(N_D = n|x_i, x_j)$$
$$= \lim_{da,db\to 0} \mathbb{P}(N_D = n|N_C = \Xi, N_B = 1, N_A = 1)$$
$$= \lim_{da,db\to 0} \frac{\mathbb{P}(N_D = n, N_C = \Xi, N_B = 1, N_A = 1)}{\mathbb{P}(N_C = \Xi, N_B = 1, N_A = 1)}$$
$$= \lim_{da,db\to 0} \frac{\mathbb{P}(N_D = n, N_{C\backslash D} = \Xi - n, N_B = 1, N_A = 1)}{\mathbb{P}(N_C = \Xi, N_B = 1, N_A = 1)}$$
$$= \frac{\mathbb{P}(N_D = n)\,\mathbb{P}(N_{C\backslash D} = \Xi - n)}{\mathbb{P}(N_C = \Xi)}$$
$$= \frac{(\lambda|D|)^n e^{-\lambda|D|}/n!}{(\lambda|C|)^{\Xi} e^{-\lambda|C|}/\Xi!} \cdot \frac{(\lambda(|C| - |D|))^{\Xi-n} e^{-\lambda(|C|-|D|)}}{(\Xi - n)!}$$
$$= \frac{\Xi!}{n!(\Xi - n)!}\left(\frac{|D|}{|C|}\right)^n \left(1 - \frac{|D|}{|C|}\right)^{\Xi-n}$$

which shows that the Ξ BSs inside the annular region $\lim_{da,db\to 0} C = \mathtt{b}(o, \|x_j\|)\backslash\mathtt{b}(o, \|x_i\|)$ make up a BPP. By letting $i = 0$, $j = L$, and defining $x_0 \triangleq o$ (i.e., $\|x_0\| = 0$), the BSs inside the circular region $\mathtt{b}(o, \|x_L\|)$ are shown to make up a BPP, thus proving Lemma 1. By letting $i = 1$ and $j = L$, Lemma 3 is proved.

B. Proof of Lemma 2

$$F_{R_1|R_L}(r|R_L) = \mathbb{P}(R_1 \le r|R_L) = 1 - \mathbb{P}(R_1 > r|R_L)$$
$$= 1 - \mathbb{P}(\min(\{x : \|x\| < \|x_L\|\}) > r|R_L)$$
$$\stackrel{(a)}{=} 1 - \prod_{\{x:\|x\|<\|x_L\|\}} \mathbb{P}(\|x\| > r|R_L)$$
$$= 1 - \prod_{\{x:\|x\|<\|x_L\|\}} \frac{\pi R_L^2 - \pi r^2}{\pi R_L^2} = 1 - \left(\frac{R_L^2 - r^2}{R_L^2}\right)^{L-1},$$

where (a) follows from Lemma 3.

C. Proof of Lemma 4

Let x be one of the BSs between x_1 and x_L in distance from the origin. Then

$$\mathbb{E}[P\|x\|^{-\alpha}|R_1, R_L] \stackrel{(a)}{=} \int_{R_1}^{R_L} Pr^{-\alpha}\frac{2r}{R_L^2 - R_1^2}\,\mathrm{d}r$$
$$= \frac{2P}{R_L^2 - R_1^2}\int_{R_1}^{R_L} r^{1-\alpha}\mathrm{d}r = \frac{2P}{R_L^2 - R_1^2}\left(\frac{r^{2-\alpha}}{2 - \alpha}\Big|_{r=R_1}^{R_L}\right)$$
$$= \frac{2P}{2 - \alpha} \cdot \frac{R_L^{2-\alpha} - R_1^{2-\alpha}}{R_L^2 - R_1^2},$$

where (a) follows from Lemma 3. Lastly, the mean of the sum of the interference from all interferers is simply the sum of their individual means, from which the result follows.

D. Proof of Lemma 5

$$\mathbb{E}[\mathcal{I}_2|R_L] = \mathbb{E}\left[\sum_{x\in\Phi\setminus\mathbb{b}(o,R_L)} P\|x\|^{-\alpha}\right]$$

$$= \lambda\int_{\mathbb{b}^c(o,R_L)} P\|x\|^{-\alpha}\mathrm{d}x = P\lambda\int_0^{2\pi}\int_{R_L}^\infty r^{-\alpha}r\ \mathrm{d}r\ \mathrm{d}\theta$$

$$= 2P\pi\lambda\int_{R_L}^\infty r^{1-\alpha}\mathrm{d}r = 2P\pi\lambda\cdot\left.\frac{r^{2-\alpha}}{2-\alpha}\right|_{r=R_L}^\infty$$

$$= \frac{2P\pi\lambda}{\alpha-2}R_L^{2-\alpha}$$

for $\alpha > 2$, and unbounded otherwise.

E. Proof of Theorem 1

Let $\sigma^2 = 0$, and evaluate at $\alpha = 4$.

$$\mathrm{P_L}(L,\beta,\gamma,\lambda) = \mathbb{P}(\mathsf{SINR}_L \geq \gamma^{-1}\beta)$$
$$= \mathbb{E}_{R_L}\left[\mathbb{E}_{R_1}\left[\mathbb{1}\left(\mathsf{SINR}_L\geq\gamma^{-1}\beta\big|R_1,R_L\right)\big|R_L\right]\right] \quad (13)$$

The SINR_L term in the above expression is

$$\mathsf{SINR}_L = \frac{PR_L^{-\alpha}}{PR_1^{-\alpha} + \mathbb{E}[\mathcal{I}_1|R_1,R_L] + \mathbb{E}[\mathcal{I}_2|R_L]}$$

$$= \frac{PR_L^{-\alpha}}{PR_1^{-\alpha} + \frac{2P(L-2)}{2-\alpha}\cdot\frac{R_L^{2-\alpha}-R_1^{2-\alpha}}{R_L^2-R_1^2} + \frac{2P\pi\lambda}{\alpha-2}R_L^{2-\alpha}}$$

$$= \frac{1}{\left(\frac{R_L}{R_1}\right)^\alpha + \frac{2(L-2)}{2-\alpha}\cdot\frac{\left(\frac{R_L}{R_1}\right)^2-\left(\frac{R_L}{R_1}\right)^\alpha}{\left(\frac{R_L}{R_1}\right)^2-1} + \frac{2\pi\lambda}{\alpha-2}R_L^2}$$

$$\stackrel{(a)}{=} \frac{1}{X^4 + (L-2)X^2 + \pi\lambda R_L^2}, \quad (14)$$

where (a) follows by defining $X = \frac{R_L}{R_1}$ and substituting $\alpha = 4$. Using (14), we now simplify $\mathsf{SINR}_L \geq \gamma^{-1}\beta$ that will then used to derive $\mathrm{P_L}(L,\beta,\gamma,\lambda)$ in (13). Defining $Y = X^2$ we have

$$\mathsf{SINR}_L \geq \gamma^{-1}\beta \stackrel{(a)}{\Rightarrow} Y^2 + (L-2)Y \leq \kappa^{-1}$$

$$\Rightarrow \left(Y+\frac{(L-2)}{2}\right)^2 \leq \kappa^{-1} + \frac{(L-2)^2}{4}$$

$$\stackrel{(b)}{\Rightarrow} 0 \leq Y \leq \sqrt{\kappa^{-1}+\frac{(L-2)^2}{4}} - \frac{(L-2)}{2}$$

$$\stackrel{(c)}{\Rightarrow} X^2 \leq \sqrt{\kappa^{-1}+\frac{(L-2)^2}{4}} - \frac{(L-2)}{2}$$

$$\stackrel{(d)}{\Rightarrow} 1 \leq X \leq \sqrt{\sqrt{\kappa^{-1}+\frac{(L-2)^2}{4}} - \frac{(L-2)}{2}}$$

$$\Rightarrow \frac{R_L}{\sqrt{\sqrt{\kappa^{-1}+\frac{(L-2)^2}{4}} - \frac{(L-2)}{2}}} \leq R_1 \leq R_L$$

where $\kappa^{-1} = \gamma/\beta - \pi\lambda R_L^2$ in (a), (b) follows from the fact that $Y \geq 1$, and (c) from $Y = X^2$. Note that (b) and (c) require $\kappa^{-1} \geq -\frac{(L-2)^2}{4}$. Step (d) follows from $X \geq 1$. The earlier condition on κ^{-1} is replaced by a more strict condition

$\kappa^{-1} \geq L-1$ in (d). Substituting this back in (13) and doing some algebraic manipulations, we get

$$\mathrm{P_L}(L,\beta,\gamma,\lambda) = \mathbb{E}_{R_L}\left[\left(\frac{R_L^2 - \frac{R_L^2}{\sqrt{\kappa^{-1}+\frac{(L-2)^2}{4}}-\frac{(L-2)}{2}}}{R_L^2}\right)^{L-1}\right]$$

$$= \mathbb{E}_{R_L}\left[\left(1 - \frac{1}{\sqrt{\gamma/\beta - \pi\lambda R_L^2 + \frac{(L-2)^2}{4}}-\frac{(L-2)}{2}}\right)^{L-1}\right],$$

from which the result follows by simply integrating across the density of R_L. Due to $\kappa^{-1} \geq L-1$ the integration limits are from 0 to $\sqrt{\frac{\gamma/\beta-(L-1)}{\pi\lambda}}$. This completes the proof.

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